

# Max $k$ -cut and the smallest eigenvalue

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## Abstract

Let  $G$  be a graph of order  $n$  and size  $m$ , and let  $\text{mc}_k(G)$  be the maximum size of a  $k$ -cut of  $G$ . It is shown that

$$\text{mc}_k(G) \leq \frac{k-1}{k} \left( m - \frac{\mu_{\min}(G) n}{2} \right),$$

where  $\mu_{\min}(G)$  is the smallest eigenvalue of the adjacency matrix of  $G$ .

An infinite class of graphs forcing equality in this bound is constructed.

**Keywords:** max  $k$ -cut; chromatic number; largest eigenvalues; largest Laplacian eigenvalue; smallest adjacency eigenvalue.

**AMS classification:** 05C50

## 1 Introduction and main results

The *maximum  $k$ -cut* of  $G$ , denoted by  $\text{mc}_k(G)$ , is the maximum number of edges in a  $k$ -partite subgraph of  $G$ . This note provides an upper bound on  $\text{mc}_k(G)$  based on  $\mu_{\min}(G)$  – the smallest eigenvalue of the adjacency matrix of  $G$ .

In [4] Mohar and Poljak gave the celebrated bound  $\text{mc}_2(G) \leq \lambda(G) n/4$ , where  $\lambda(G)$  is the maximum eigenvalue of the Laplacian matrix of  $G$ . However, one may question how fit  $\lambda(G)$  is for such a bound on  $\text{mc}_k(G)$ , since  $\text{mc}_k(G)$  is a Lipschitz function in the number of edges  $m$ , whereas  $\lambda(G)$  may be quite volatile in  $m/n$ . Indeed, raising the degree of a single vertex of maximum degree  $\Delta(G)$  in  $G$  can raise  $\lambda(G)$  accordingly, due to the inequality  $\lambda(G) > \Delta(G)$ . In contrast,  $\mu_{\min}(G)$  depends more robustly on  $m/n$ , and hence may be a better choice than  $\lambda(G)$  for upper bounds on  $\text{mc}_2(G)$ . In [6], Trevisan came to grips with similar problems, but the emphasis of his work is on algorithms and no bound was produced in closed form. Thus, we propose the following theorem:

**Theorem 1** *If  $G$  is a graph with  $n$  vertices and  $m$  edges, then*

$$\text{mc}_k(G) \leq \frac{k-1}{k} \left( m - \frac{\mu_{\min}(G) n}{2} \right). \quad (1)$$

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**Proof** Let  $G$  be as required and suppose that its vertex set is  $[n] := \{1, \dots, n\}$ . Let  $H$  be a  $k$ -partite subgraph of  $G$  with  $\text{mc}_k(G)$  edges, and let  $[n] = V_1 \cup \dots \cup V_k$  be the partition of the vertices of  $H$  into  $k$  edgeless sets. The idea of the proof is to use Rayleigh's principle to construct  $k$  upper bounds on  $\mu_{\min}(G)$ , and then take their average as an upper bound on  $\mu_{\min}(G)$ .

For each  $i \in [k]$ , define a vector  $\mathbf{y}^{(i)} := (y_1^{(i)}, \dots, y_n^{(i)})$  as

$$y_j^{(i)} := \begin{cases} -k+1, & \text{if } j \in V_i, \\ 1, & \text{if } j \in [n] \setminus V_i. \end{cases}$$

Write  $\langle \mathbf{u}, \mathbf{v} \rangle$  for the inner product of the vectors  $\mathbf{u}$  and  $\mathbf{v}$ , and note that for each  $i \in [k]$ , Rayleigh's principle implies that

$$\mu_{\min}(G) \left\| \mathbf{y}^{(i)} \right\|^2 \leq \langle A\mathbf{y}^{(i)}, \mathbf{y}^{(i)} \rangle.$$

Hence, summing these inequalities for all  $i \in [k]$ , we get

$$\mu_{\min}(G) \sum_{i \in [k]} \left\| \mathbf{y}^{(i)} \right\|^2 \leq \sum_{i \in [k]} \langle A\mathbf{y}^{(i)}, \mathbf{y}^{(i)} \rangle. \quad (2)$$

On the one hand, for  $\sum_{i \in [k]} \left\| \mathbf{y}^{(i)} \right\|^2$  we have

$$\sum_{i \in [k]} \left\| \mathbf{y}^{(i)} \right\|^2 = \sum_{i \in [k]} \left( (k-1)^2 |V_i| + n - |V_i| \right) = (k^2 - k) n. \quad (3)$$

On the other hand, writing  $e(X)$  for the number of edges induced by a set  $X$  and  $e(X, Y)$  for the number of cross-edges between the sets  $X$  and  $Y$ , for every  $i \in [k]$ , we see that

$$\langle A\mathbf{y}^{(i)}, \mathbf{y}^{(i)} \rangle = 2(k-1)^2 e(V_i) + \sum_{j \in [k] \setminus \{i\}} 2e(V_j) - \sum_{j \in [k] \setminus \{i\}} 2(k-1)e(V_i, V_j) + \sum_{j, l \in [k] \setminus \{i\}, j \neq l} 2e(V_l, V_j).$$

Summing these inequalities for all  $i \in [k]$ , we get four terms in the right side:

$$\begin{aligned} \sum_{i \in [k]} 2(k-1)^2 e(V_i) &= 2(k-1)^2 (m - \text{mc}_k), \\ \sum_{i \in [k]} \sum_{j \in [k] \setminus \{i\}} 2e(V_j) &= 2(k-1)(m - \text{mc}_k), \\ - \sum_{i \in [k]} \sum_{j \in [k] \setminus \{i\}} 2(k-1)e(V_i, V_j) &= -4(k-1)\text{mc}_k, \\ \sum_{i \in [k]} \sum_{j, l \in [k] \setminus \{i\}, j \neq l} 2e(V_l, V_j) &= 2(k-2)\text{mc}_k. \end{aligned}$$

Hence, for  $\sum_{i \in [k]} \langle A\mathbf{y}^{(i)}, \mathbf{y}^{(i)} \rangle$  we obtain

$$\begin{aligned} \sum_{i \in [k]} \langle A\mathbf{y}^{(i)}, \mathbf{y}^{(i)} \rangle &= 2(k-1)^2(m - \text{mc}_k) + 2(k-1)(m - \text{mc}_k) - 4(k-1)\text{mc}_k + 2(k-2)\text{mc}_k \\ &= 2k(k-1)(m - \text{mc}_k) - 2k\text{mc}_k = 2k(k-1) \left( m - \frac{k}{k-1}\text{mc}_k \right). \end{aligned}$$

Finally, combining the last equality with (2) and (3), we get

$$\frac{\mu_{\min}(G)n}{2} \leq m - \frac{k}{k-1}\text{mc}_k,$$

completing the proof of (1).  $\square$

Note that the above proof also applies to weighted graphs, i.e., graphs whose edges have been assigned positive real numbers. For  $k = 2$ , inequality (1) can be obtained from Lemma 1 of Delorme and Poljak [2] by letting  $u = \lfloor 2m/n - d_i \rfloor$ , where  $d_1, \dots, d_n$  are the degrees of  $G$ <sup>1</sup>. Likewise, (1) can also be obtained by letting  $d = \lfloor 2m/n - d_i \rfloor$  in equation (9) of the paper of van Dam and Sotirov [1].

Let us note that equality may hold in (1) for numerous graphs, both regular and irregular. Our next goal is to exhibit an infinite class of such graphs, for which we need some preparation.

Suppose that  $r \geq k \geq 2$  and write  $t_k(n)$  for the maximum number of edges in a  $k$ -partite graph of order  $n$ . The numbers  $t_k(n)$  are called Turán numbers, and it is known that

$$t_k(n) = \frac{k-1}{2k} (n^2 - s^2) + \binom{s}{2},$$

where  $s$  is the remainder  $n \bmod k$ . It is not hard to see that

$$\frac{k-1}{2k}n^2 - \frac{k}{8} \leq t_k(n) \leq \frac{k-1}{2k}n^2.$$

Equality on the right holds if and only if  $k$  divides  $n$ . Equality on the left holds if and only if  $k$  is even and  $n = k/2 \bmod k$ .

The sum of all weights of a weighted graph is called its *total weight*, and the maximum  $k$ -cut of a weighted graph is the maximum total weight of its  $k$ -partite subgraphs. Next, we give a lower bound on  $\text{mc}_k(G)$  that may well be known.

**Theorem 2** *Let  $r \geq k \geq 2$ . If  $G$  is a weighted  $r$ -partite graph with total weight  $m$ , then*

$$\text{mc}_k(G) \geq \frac{t_k(r)}{\binom{r}{2}} m.$$

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<sup>1</sup>Lovász stated this fact in Proposition 6.4.4 [3] without details, and Trevisan missed his point in the footnote on p. 1772 of [6].

**Proof** Let  $K$  be the weighted complete graph of order  $r$ , whose vertices are the vertex classes of  $G$ , and the edge weights are the sums of the weights of all edges across the corresponding classes. Clearly the total weight of  $K$  is  $m$ . Define a random variable  $\mathbf{X}_k(K)$  equal to the total weight of a randomly chosen  $k$ -partite subgraph of  $K$  with  $t_k(r)$  edges. Let  $M$  be the number of all such subgraphs of  $K$ . By symmetry, each edge of  $K$  belongs to the same number of such subgraphs, which obviously is

$$\frac{t_k(r)}{\binom{r}{2}} M.$$

Therefore,

$$\mathbb{E}(\mathbf{X}_k(K)) = \frac{1}{M} \cdot \frac{t_k(r)}{\binom{r}{2}} M m = \frac{t_k(r)}{\binom{r}{2}} m.$$

Thus, there is a  $k$ -partite subgraph of  $G$  of total weight at least  $t_k(r) m / \binom{r}{2}$ , as claimed.  $\square$

Note that Theorem 2 is an improvement over the straightforward lower bound  $\text{mc}_k(G) \geq (1 - 1/k) m$ .

We are now ready to describe a class of regular graphs that force equality in (1).

Let  $\chi \geq k \geq 2$  and suppose that  $k$  divides  $\chi$ . Take a  $t$ -regular graph  $H$  of order  $n$  satisfying  $\omega(H) \geq \chi$  and

$$|\mu_{\min}(H)| < \frac{t}{\chi - 1}.$$

Let  $J_\chi$  be the  $\chi \times \chi$  matrix of all-ones and  $I_\chi$  be the identity matrix of order  $\chi$ . Write  $A(H)$  for the adjacency matrix of  $H$ . The Kronecker product  $B := (J_\chi - I_\chi) \otimes A(H)$  is a symmetric  $(0, 1)$ -matrix with zero diagonal. Let  $G$  be the graph with adjacency matrix  $B$ . Clearly,  $G$  is  $(\chi - 1)t$ -regular graph of order  $\chi n$ . Also,

$$\mu_{\min}(G) = \min\{-t, (\chi - 1)\mu_{\min}(H)\} = -t.$$

Using the fact that  $\omega(H) \geq \chi$ , one can show that  $\omega(G) = \chi$ , which obviously implies that  $\chi(G) = \chi$  as well. Since  $k$  divides  $\chi$ , we have  $t_k(\chi) = \frac{k-1}{2k}\chi^2$  and Theorem 2 implies that

$$\begin{aligned} \text{mc}_k(G) &\geq \frac{\frac{k-1}{2k}\chi^2}{\frac{\chi(\chi-1)}{2}} e(G) = \frac{k-1}{k} \frac{\chi}{\chi-1} e(G) = \frac{k-1}{k} e(G) + \frac{k-1}{k} \frac{t(\chi-1)\chi n}{2(\chi-1)} \\ &= \frac{k-1}{k} \left( e(G) + \frac{tv(G)}{2} \right) = \frac{k-1}{k} \left( e(G) - \frac{\mu_{\min}(G)v(G)}{2} \right). \end{aligned}$$

Hence the graph  $G$  forces equality in (1).

To conclude, we show that two results of the recent paper [1] are simple consequences of a result proved in [5]. In [1], van Dam and Sotirov showed that

$$\text{mc}_k(G) \leq \frac{n(k-1)}{2k} \lambda(G), \quad (4)$$

where  $\lambda(G)$  is the maximum eigenvalue of the Laplacian matrix of  $G$ . However, (4) follows immediately from an inequality in [5] that reads as:

If  $H$  is a  $k$ -partite graph and  $\mu(H)$  is the maximum eigenvalue of its adjacency matrix, then

$$\mu(H) \leq \frac{k-1}{k} \lambda(H). \quad (5)$$

Indeed, if  $H$  is a  $k$ -partite subgraph of  $G$  with  $\text{mc}_k(G)$  edges, then

$$\frac{2\text{mc}_k(G)}{n} \leq \mu(H) \leq \frac{k-1}{k} \lambda(H) \leq \frac{k-1}{k} \lambda(G),$$

and inequality (4) follows. Note that for regular graphs (4) and (1) are equivalent, but they are incomparable in general.

Further, van Dam and Sotirov show that if  $G$  has  $m$  edges, then its chromatic number  $\chi(G)$  satisfies:

$$\chi(G) \geq 1 + \frac{2m}{n\lambda(G) - 2m}. \quad (6)$$

However, this inequality is also a simple consequence of (5). Indeed, rewriting (5) as

$$\chi(G) \geq 1 + \frac{\mu(G)}{\lambda(G) - \mu(G)},$$

inequality (6) follows as

$$\chi(G) \geq 1 + \frac{\mu(G)}{\lambda(G) - \mu(G)} \geq 1 + \frac{2m/n}{\lambda(G) - 2m/n} = 1 + \frac{2m}{n\lambda(G) - 2m}.$$

## References

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